AV – 82$5

M. A/M. Sc III nd Sem. 2015-16

Mathematics

Fuzzy Sets & Their Applications - I

(Suggested Solution).

1 (i) Let X be any set and $\mathcal{F}([0,1])$ denote the set of all ordinary fuzzy sets defined on the universal set $[0,1]$. A function $A : X \longrightarrow \mathcal{F}([0,1])$ is said to be a fuzzy set of type 2.

(ii) $|A| = .13 + .27 + .53 + .93 + 1 = 2.86$

So scalar cardinality of A is 2.86.

(iii) Let $x \in {}^{\beta+}A$ then $A(x) > \beta \geq \alpha$ so $A(x) > \alpha$ and so $x \in {}^{\alpha+}A$. Thus ${}^{\alpha+}A \supseteq {}^{\beta+}A$.

(iv) Let X be ~~and~~ any set and $A \in \mathcal{F}(X)$ (the fuzzy power set of X) then

$$A = \bigcup_{\alpha \in \Lambda(A)} {}_{\alpha}A$$ where $\Lambda(A)$ is the level set of A, $\cup$ denotes the standard fuzzy union and ${}_{\alpha}A$ is the fuzzy set defined by ${}_{\alpha}A(x) = \alpha \cdot {}^{\alpha}A(x)$ & ${}^{\alpha}A(x) = 1 \; \forall \; x \in {}^{\alpha}A$.

(v) $f^{-1}(B)$ is defined as a fuzzy set on X whose membership function is given by

$$f^{-1}(B)(x) = B(f(x)).$$

(vi) Student's choice.

(vii) Any membership grade $A_a \in [0,1]$ is said to be a dual point of a if it satisfied the

(viii) If A is any fuzzy number then by virtue of definition of fuzzy number each α-cut ${}^{\alpha}A$ of A ($\alpha \in (0, 1]$) is a closed interval which is a convex set, it follows that A is a convex fuzzy set.

(ix) Let $R(X, X)$ and $Q(Y, Y)$ denote two binary relations (on X and Y respectively) and $h : X \to Y$ be a mapping. f is said to be a strong homomorphism if

(a) $\langle x_1, x_2 \rangle \in R \Rightarrow \langle h(x_1), h(x_2) \rangle \in Q$ $\forall x_1, x_2 \in X$

& (b) $\langle y_1, y_2 \rangle \in Q \Rightarrow \langle x_1, x_2 \rangle \in R$ $\forall y_1, y_2 \in Y$ where $x_1 \in h^{-1}(y_1)$ and $x_2 \in h^{-1}(y_2)$.

(x) $$w_i(0, 1) = \sup \{x \in [0,1] \mid i(0, x) \leq 1\}$$
$$= \sup \{x \in [0,1] \mid 0 \leq 1\}$$
$$= 1.$$

2 (a). The student is supposed to ~~tea~~ list and describe important features of the paradigm shift introduced by fuzzy sets. A few features are listed below:

(i) It allowes measurement uncertainities and observational errors.

(ii) Capability of managing complexity and controlling computational cost.

(iii) Greater expressive power

(iv) Use of natural language

(v) Human friendlyness.

2 (b) $\because$ $\forall\ x \in X$ we have

$$\max(A(x), B(x)) = A(x) + B(x) - \min(A(x), B(x))$$

we ~~cat~~ conclude that

$$(A \cup B)(x) = A(x) + B(x) - (A \cap B)(x).$$

Hence $|A \cup B| = |A| + |B| - |A \cap B|$.

3 (a). (i) As $x \in {}^{\alpha+}(A \cap B) \iff (A \cap B)(x) > \alpha$

$\iff \min(A(x), B(x)) > \alpha$

$\iff A(x) > \alpha$ and $B(x) > \alpha$

$\iff x \in {}^{\alpha+}A$ and $x \in {}^{\alpha+}B$

$\iff x \in {}^{\alpha+}A \cap {}^{\alpha+}B$

while $x \in {}^{\alpha+}(A \cup B) \iff (A \cup B)(x) > \alpha$

$\iff \max(A(x), B(x)) > \alpha$

$\iff A(x) > \alpha$ or $B(x) > \alpha$

$\iff x \in {}^{\alpha+}A$ or $x \in {}^{\alpha+}B$

$\iff x \in ({}^{\alpha+}A \cup {}^{\alpha+}B)$

the results follow.

(ii) Students may compute ${}^{\alpha}(\bar{A})$ and show that it is not equal to $\overline{{}^{\alpha}(A)}$.

3 (b) For every $A \in \mathcal{F}(X)$ $\quad A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}A$

where ${}_{\alpha}A$ is a fuzzy set defined by

$${}_{\alpha}A(x) = \alpha \cdot {}^{\alpha}A(x) \quad \text{and} \quad {}^{\alpha}A(x) = 1 \;\forall x \in {}^{\alpha}A.$$

Here $\cup$ stands for standard fuzzy union.

Proof: Choose any $x \in X$ and let $A(x) = a$. Then

$$\left(\bigcup_{\alpha \in [0,1]} {}_{\alpha}A\right)(x) = \sup_{\alpha \in [0,1]} {}_{\alpha}A(x)$$

$$= \max\left[\sup_{\alpha \in [0,a]} {}_{\alpha}A(x),\ \sup_{\alpha \in (a,1]} {}_{\alpha}A(x)\right]$$

Now for each $\alpha \in (a, 1]$ we have $A(x) = a < \alpha$ and therefore ${}_{\alpha}A(x) = 0$. On the other hand for each $\alpha \in [0, a]$ we have $A(x) = a \geq \alpha$ and so ${}_{\alpha}A(x) = \alpha$.

Hence

$$\left(\bigcup_{\alpha \in [0,1]} {}_{\alpha}A\right)(x) = \sup_{\alpha \in [0,a]} \alpha = a = A(x)$$

As this argument holds for each $x \in X$ we conclude the result.

4. (i) If $f^{-1}(y) = \phi$ then $f\left(\bigcup_{i \in I} A_i\right)(y) = 0$ and $\left(\bigcup f A_i\right)(y) = 0$ so the result follows in this case.

Now if $f^{-1}(y) \neq \phi$ then

$$f\left(\bigcup_{i \in I} A_i\right)(y) = \sup_{x | f(x) = y} \left(\left(\bigcup A_i\right)(x)\right) = \sup_{x | f(x) = y} \left(\sup_{i \in I} (A_i(x))\right)$$

$$= \sup_{i \in I} \left(\sup_{x \mid f(x)=y} A_i(x) \right)$$

$$= \bigcup_{i \in I} \left(\sup_{x \mid f(x)=y} A_i(x) \right) = \bigcup \left(f(A_i) \right)(y)$$

Hence $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$.

(ii) We have

$$\inf_{i \in I} A_i(x) \leq A_j(x) \qquad \forall\, j \in I$$

so $\sup_{x \in X \mid f(x)=y} \left(\inf_{i \in I} A_i(x) \right) \leq \sup_{x \in X \mid f(x)=y} \left(A_j(x) \right)$ for each $y \in Y$

i.e. $\sup_{x \in X \mid f(x)=y} \left(\left(\bigcap_{i \in I} A_i \right)(x) \right) \leq \sup_{x \in X \mid f(x)=y} \left(A_j(x) \right)$ for each $y \in Y$

i.e. $f\left(\bigcap_{i \in I} A_i\right)(y) \leq \left(f(A_j) \right)(y)$ for each $y \in Y$

i.e. $f\left(\bigcap_{i \in I} A_i\right)(y) \leq \inf_{j \in J} f(A_j)(y)$ for each $y \in Y$

i.e. $f\left(\bigcap_{i \in I} A_i\right)(y) \leq \bigcap_{j \in J} f(A_j)(y)$ for each $y \in Y$

Hence $f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$.

(iii) For each $x \in X$ we have

$$\left(f^{-1}\left(\bigcup_{i \in I} B_i \right) \right)(x) = \left(\bigcup_{i \in I} B_i \right) f(x)$$

$$= \sup \left(B_i\, f(x) \right)$$

$$= \sup_{i \in I} \left(f^{-1}(B_i)(x) \right)$$

$$= \left(\bigcup_{i \in I} f^{-1}(B_i) \right)(x)$$

Hence $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$.

(iv) For each $x \in X$ we have

$$f^{-1}\left(\bigcap_{i \in I} B_i\right)(x) = \left(\bigcap B_i\right) f(x)$$

$$= \inf_{i \in I} B_i(f(x))$$

$$= \inf_{i \in I} \left(f^{-1}(B_i) \right)(x)$$

$$= \left(\bigcap_{i \in I} f^{-1}(B_i) \right)(x)$$

Hence $f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$.

(5) Statement : Let c be a function from $[0,1]$ to $[0,1]$. Then c is a fuzzy complement (involutive) if and only if there exists a continuous function $g : [0,1] \longrightarrow \mathbb{R}$ (the set of real numbers) such that $g(0) = 0$ g is strictly increasing and

$$c(a) = g^{-1}\left(g(1) - g(a) \right) \quad \forall a \in [0,1] \qquad (1)$$

Proof : First we assume that the function g is given that satisfies the mentioned conditions, and ~~shot~~ show that c satisfies

$c(1) = 0$

(ii) $\forall\ a, b \in [0,1] \quad a \le b \Rightarrow c(a) \ge c(b)$

(iii) c is a continuous function

(iv) c is involutive i.e. $c(c(a)) = a \quad \forall\ a \in [0,1]$

In fact we need to show (ii) and (iv) hold so that using the result that any function c that satisfies (ii) & (iv) ~~cont~~ above must satisfy (1) & (iii) also we conclude this part.

We ~~can~~ now construct pseudoinverse of g denoted by $g^{(-1)}$ as a function from $\mathbb{R}$ to $[0,1]$ given by

$$g^{(-1)}(a) = \begin{cases} 0 & \text{for } a \in (-\infty, 0) \\ g^{-1}(a) & \text{for } a \in [0, g(1)] \\ 1 & \text{for } a \in (g(1), \infty) \end{cases}$$

where g^{-1} is the ordinary inverse of g from $[0, g(1)]$ to $[0,1]$. (For this part g^{-1} exists as g is strictly increasing). Let $c(a) = g^{-1}(g(1) - g(a))$ (as given).

For $a, b \in [0,1]$ if $a < b$ then $g(a) < g(b)$ (as g is strictly increasing) so $g(1) - g(a) > g(1) - g(b)$ & consequently

$$c(a) = g^{-1}[g(1) - g(a)] > g^{-1}[g(1) - g(b)] = c(b)$$

i.e. $c(a) > c(b)$ This shows (ii)

Next by show (iv). For any $a \in [0,1]$

$$c(c(a)) = g^{-1}[g(1) - g(c(a))] = g^{-1}[g(1) - g(g^{-1}(g(1) - g(a)))]$$

$$= g^{-1}[g(1) - g(1) + g(a)] = g^{-1}(g(a)) = a$$

Now we prove the converse part. To start with, we mention that " A continuous fuzzy involutive complement has one and only one equilibrium point." Let e_c be the equilibrium point of c then $e_c \in (0, 1)$. Let $h : [0, e_c] \longrightarrow [0, b]$ be given by $h(a) = \frac{ba}{e_c}$ then h is continuous strictly increasing bijective map such that $h(0) = 0$ and $h(e_c) = b$ where b is any fixed real numbers

Define $g : [0, 1] \longrightarrow \mathbb{R}$ by

$$g(a) = \begin{cases} h(a) & \text{if } a \in [0, e_c] \\ 2b - h(c(a)) & \text{if } a \in (e_c, 1] \end{cases}$$

Then $g(0) = h(0) = 0$. Since c and h are continuous it follows that g is continuous and as h is strictly increasing it follows that g is also strictly increasing Next we observe that the following function works as pseudoinverse for g.

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ h^{-1}(a) & \text{if } a \in [0, b] \\ c(h^{-1}(2(b) - a) & \text{if } a \in [b, 2b] \\ 1 & \text{if } a \in (2b, \infty) \end{cases}$$

Finally we show that with this set up equation (1) given in the statement of the theorem holds.

For any $a \in [0, 1]$ if $a \in [0, e_c]$ then $g^{-1}(g(1) - g(a))$

$= g^{-1}[g(1) - h(a)] = g^{-1}[2b - h(a)] = c(h^{-1}[2b - (2b - h(a))])$

$= c(a)$ while if $a \in (e_c, 1]$ then $g^{-1}[g(1) - g(a)]$

$= g^{-1}[2b - (2b - h(c(a)))] = g^{-1}[h(c(a))] = h^{-1}[h(c(a))]$

Que 6. To prove the result we need to show that for some $w > 0$ (if required under limiting conditions) $i_w(a,b)$ equals to $i_{min}(a,b)$ and also for some $w > 0$ $i_w(a,b)$ equals to $\min(a,b)$. First we find w such that $i_w(a,b) = i_{min}(a,b)$. Since

$$i_w(a,b) = 1 - \min\left(1, [(1-a)^w + (1-b)^w]^{\frac{1}{w}}\right) \quad (1)$$

we see that when $a = 1$ $i_w(1,b) = b$ and when $b = 1$ $i_w(a,1) = a$ independent of the choice of w. Also when neither $a = 1$ nor $b = 1$ we have for $w \to 0$

$$\cancel{\lim_{w \to 0}} \; [(1-a)^w + (1-b)^w]^{1/w} \longrightarrow \infty \quad \text{hence}$$

$$\lim_{w \to 0} i_w(a,b) = 0. \quad \forall\, a, b \in [0,1).$$

Nowe we proceed for the second part. To start we show $\lim_{w \to \infty} \min\left[1, (a^w + b^w)^{\frac{1}{w}}\right] = \max(a,b) \quad \forall\, a,b \in [0,1]$

This result is clear when $a = 0$ or $b = 0$ and when $a = b$. ~~A~~ So let $a \neq 0$ $b \neq 0$ & $a \neq b$ and $\min\left[1, (a^w + b^w)^{\frac{1}{w}}\right] = (a^w + b^w)^{\frac{1}{w}}$ then we need to show that

$$\lim_{w \to \infty} (a^w + b^w)^{\frac{1}{w}} = \max(a,b)$$

Since $a \neq b$ we may assume first that $a < b$ (the part $b < a$ ~~wi~~ can be proved on similar lines).

Set $Q = (a^w + b^w)^w$. Then

$$\lim_{w\to\infty} \log Q = \lim_{w\to\infty} \frac{\log(a^w+b^w)}{w}.$$ Using L'Hospital's rule,

we obtain

$$\lim_{w\to\infty} \log Q = \lim_{w\to\infty} \frac{a^w\log a + b^w\log b}{a^w+b^w}$$

$$= \lim_{w\to\infty} \frac{\left(\frac{a}{b}\right)^w\log a + \log b}{\left(\frac{a}{b}\right)^w + 1} = \log b$$

Hence $\lim_{w\to\infty} Q = \lim_{w\to\infty} (a^w+b^w)^{\frac{1}{w}} = b = \max(a,b)$

It still remains to be shown that the result is true when min is 1. In this case

$$(a^w+b^w)^{\frac{1}{w}} \geq 1$$

or $\quad a^w+b^w \geq 1 \qquad \forall\ w\in(0,\infty).$

When $w\to\infty$ the last inequality holds if $a=1$ or $b=1$

Hence

$$\lim_{w\to\infty} \min\left[1, (a^w+b^w)^{\frac{1}{w}}\right] = \max(a,b)$$

$$\forall\ a,b\in[0\ 1]$$

Now as $a,b\in[0,1]$ $\ 1-a, 1-b\in[0,1]$ & so we have

$$\lim_{w\to\infty} \min\left[1, \{(1-a)^w+(1-b)^w\}^{\frac{1}{w}}\right] = \max[1-a, 1-b]$$

Hence $$\lim_{w\to\infty} i_w(a,b) = 1 - \lim_{w\to\infty} \min\left[1, \{(1-a)^w+(1-b)^w\}^{\frac{1}{w}}\right]$$

$$= 1 - \max[1-a, 1-b]$$

$$= \min(a, b)$$

Que 7. (i) We have

$$MIN\,[A, MIN(B,C)]\,z = \sup_{z=\min(x,y)} \min\,[A(x), MIN(B,C)(y)]$$

$$= \sup_{z=\min(x,y)} \min\left[A(x), \sup_{y=\min(u,v)} \min[B(u), C(v)]\right]$$

$$= \sup_{z=\min(x,y)} \sup_{y=\min(u,v)} \min\,[A(x), B(u), C(v)]$$

$$= \sup_{z=\min(x,u,v)} \min\,[A(x), B(u), C(v)]$$

$$= \sup_{z=\min(s,v)} \sup_{s=\min(x,u)} \min\,[A(x), B(u), C(v)]$$

$$= \sup_{z=\min(s,v)} \min\left\{\left[\sup_{s=\min(x,u)} \min[A(x), B(u)], C(v)\right]\right]$$

$$= \sup_{z=\min(s,v)} \min\,[MIN(A,B)(s), C(v)]$$

$$= MIN\,[MIN(A,B), C\,](z)$$

$$\forall\, z \in \mathbb{R}$$

Hence MIN operation is associative. The proof for MAX operation is similar [$z = \min(x,y)$ etc ~~are~~ need to be replaced by $z = \max(x,y)$ etc. only].

(ii) $\forall\ z \in \mathbb{R}$ we have

$$\mathrm{MIN}\,[A,\ \mathrm{MAX}(A,B)]\,(z) = \sup_{z=\min(x,y)} \min\,[A(x)\ \ \mathrm{MAX}(A,B)(y)]$$

$$= \sup_{z=\min(x,y)} \min \left[A(x),\ \sup_{y=\max(u,v)} \min\,[A(u), B(v)]\right]$$

$$= \sup_{z=\min(x,\,\max(u,v))} \min\,[A(x),\ A(u),\ B(v)].$$

Let M denote the right hand side of the last equation.

Since B is a fuzzy number, $\exists\ v_0 \in \mathbb{R}$ such that $B(v_0) = 1$. By $z = \min[z,\ \max(z, v_0)]$ we have

$$M \geq \min\,[A(z),\ A(z),\ B(v_0)] = A(z)$$

On the other hand since $z = \min\,[x,\ \max(u,v)]$, we have

$$\min(x, u) \leq z \leq x \leq \max(x, u).$$

As fuzzy numbers are convex fuzzy sets we have

$$A(z) \geq \min\,[A[\min(x,u)],\ A\,[\max(x,u)]]$$

$$= \min\,[A(x)\ A(u)]$$

$$\geq \min\,[A(x),\ A(u),\ B(v)]$$

Thus $M = A(z)$ and consequently $\mathrm{MIN}\,[A,\ \mathrm{MAX}(B,C)] = A$. The proof of the other absorption law is similar.

Que 8 (a) To find transitive max-min closure for the given fuzzy relation $R(X, X)$, we will use the following algorithm with 3 steps

1 $R' = R \cup (R \circ R)$

2 If $R' \neq R$ make $R = R'$ and go to step 1

3 Stop if $R' = R$ to get $R_T = R'$

($R \circ R$ operation uses max min composition)

A routine and simple calculation shows that after 3 iterations $R_T = R'$ is given by

$$\begin{bmatrix} .7 & .5 & .5 & .5 \\ 0 & .4 & .8 & 1 \\ 0 & .4 & .4 & .4 \\ 0 & .4 & .8 & .4 \end{bmatrix}$$

(b) Here $R_{T(i)}$ stands for i-transitive closure of the fuzzy relation R on X^2, and inductively $R^{(n)}$ is defined as $R^{(n)} = R \overset{i}{\circ} R^{(n-1)}$ with $R^{(1)} = R$.

We know that for this composition the fuzzy relation E on X^2 defined by

$$E(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

acts as identity. So reflexivity of R gives $E \subseteq R$ [R is reflexive means $R(x, x) = 1 \ \forall x \in X$].

Now $R = E \overset{i}{\circ} R \subseteq R \overset{i}{\circ} R = R^{(2)}$. Thus $R^{(m)} \subseteq R^{(m+1)}$.

$R^{(n-1)}$ (n)

For any $x, y \in X$ if $x = y$ then $R^{(n)}(x,x) = R^{(n-1)}(x,x) = 1$ while if $x \neq y$ then

$$R^{(n)}(x,y) = \sup_{z_1, z_2, \dots, z_{n-1}} i\,[R(x,z_1), R(z_1, z_2), \dots R(z_{n-1}, y)]$$

Since $|X| = n$ the sequence $x = z_0, z_1, \dots, z_{n-1}, z_n = y$ of $n+1$ elements must contain at least two equal elements. Assume $z_r = z_s$ where $r < s$ then

$$i\,[R(x,z_1), \dots, R(z_{r-1}, z_r) \dots R(z_s, z_{s+1}) \dots; R(z_{n-1}, y)]$$

$$\leq i\,[R(x,z_1) \dots R(z_{r-1}, z_r), R(z_s, z_{s+1}), \dots, R(z_{n-1}, y)]$$

$$\leq R^{(k)}(x,y) \qquad k \leq n-1$$

$$\leq \cancel{R^{(n-1)}(x,z} \; R^{(n-1)}(x,y)$$

Hence $\forall\, x, y \in X \quad R^{(n)}(x,y) \leq R^{(n-1)}(x,y)$

& consequently $R^{(n)} \subseteq R^{(n-1)}$. It follows that $R^{(n)} = R^{(n-1)}$ and, therefore, $R_{T(i)} = R^{(n-1)}$ as desired.